LUSZTIG'S GEOMETRIC CONSTRUCTION OF THE CANONICAL BASIS PART I - THE HALL CATEGORY OF A QUIVER

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Abstract. These are notes prepared for a 30 minute talk with the same title given at the European Network for Representation Theory Summer School on *Quiver Hecke Algebras* at the *Institute d'Etudes Scientifiques de Cargese*, June 17-26, 2014. The main source of these notes is [Sch09]. Most results are due to [Lus90] who also gives a detailed exposition of this material in [Lus10].

1. Introduction

This talk brings together many techniques and results developed in the previous talks of the school to present a geometric construction of $U_\nu^{\mathbb{Z}}(\mathfrak{b}^+)$ and it's canonical $\mathbb{Z}[\nu,\nu^{-1}]$-basis.

We start with an oriented quiver Q (with vertices $Q_0=\{1,\ldots,n\}$ and edges $h\in Q_1$). Quivers have already served to define Hecke algebras and quantum groups (via Dynkin diagrams). We will now define the geometric *Hall category* $\mathcal{H}_Q$ build from $\mathbf{Rep}^{\mathrm{fd}}(Q)$. It is freely generated by some simple perverse sheaves which are G-equivariant.

In Part II (by Huijun Zhao), we will first see some examples for finite type quivers and then present the main result that $K_0(\mathcal{H}_Q)$ is isomorphic to $U_\nu^{\mathbb{Z}}(\mathfrak{b}^+)$ and that classes of simple perverse sheaves give a canonical basis in a geometric way. These results are due to [Lus90].

2. Moduli spaces associated to quivers

In this talk, we will work over $k=\mathbb{C}$. How can we turn $\mathbf{Rep}^{\mathrm{fd}}(Q)$ into a moduli space? Let $\alpha\in\mathbb{N}^I$ be a dimension vector. Consider

$$E_\alpha=\oplus_{h\in Q_1}\mathrm{Hom}_k(k^{\alpha_{s(h)}},k^{\alpha_{t(h)}})=\text{“}Q\text{-Representations on }\oplus_{i\in I}k^{\alpha_i}\text{”}$$

as an algebraic variety of dimension $\sum_{h\in Q_1}\alpha_{s(h)}\alpha_{t(h)}$. To obtain Q-representations up to isomorphism consider

$$\mathcal{M}_\alpha:={E_\alpha}\big/{G_\alpha},\qquad \text{where } G_\alpha:=\prod_{i\in I}\mathrm{GL}(\alpha_i,k).$$

Remark 2.1. $\mathcal{M}_\alpha$ is a quotient stack. We can however ignore this technicality here as we are interested in (derived) categories of constructible sheaves on $\mathcal{M}_\alpha$ which can be described as G_α-equivariant sheaves on E_α.

Further, consider the moduli space of flags of representations and subrepresentations of Q: For α, $\beta\in\mathbb{N}^I$, set

$$E_{\alpha,\ldots,\alpha_n}:=\{N_n\leq N_{n-1}\leq\ldots\leq N_0 \mid \text{submodule chains in }\mathbf{Rep}^{\mathrm{fd}}(Q), N_i/N_{i+1}\in E_{\alpha_i}\}.$$

$G_{\alpha_1+\ldots+\alpha_n}$ acts naturally on $E_{\alpha_1,\ldots,\alpha_n}$ and we can define another quotient stack by

$$\mathcal{E}_{\alpha_1,\ldots,\alpha_n}:={E_{\alpha_1,\ldots,\alpha_n}}\Big/{G_{\alpha_1+\ldots+\alpha_n}}.$$

There is an obvious forgetful map

$$q\colon E_{\alpha,\ldots,\alpha_n}\to E_{\alpha_1+\ldots+\alpha_n}$$

remembering only N_0 of the flag, which is proper (fibres are (projective) flag varieties) and factors through the quotients

$$q=q_{\alpha_1,\ldots,\alpha_n}\colon\mathcal{E}_{\alpha_1,\ldots,\alpha_n}\to\mathcal{M}_{\alpha_1+\ldots\alpha_n}.$$

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We will consider the correspondence

$$\begin{tikzcd} & \mathcal{E}_{\alpha,\beta} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{M}_\alpha \times \mathcal{M}_\beta & & \mathcal{M}_{\alpha+\beta}, \end{tikzcd}$$

where $p(N_1 \le N_0) = (N_0/N_1, N_1)$ and $q(N_1 \le N_0) = N_0$ as above. Note that p is smooth.

3. The Hall category and simple perverse sheaves

Let $D^b(\mathcal{M}_\alpha)$ denote the bounded derived category of G_α-equivariant k-constructible complexes over E_α. Using the above correspondence, we obtain a functor

$$\begin{aligned} m\colon D^b(\mathcal{M}_\alpha \times \mathcal{M}_\beta) &\to D^b(\mathcal{M}_{\alpha+\beta}), \\ F &\mapsto q_! p^*(F)[\dim p], \end{aligned}$$

called *induction.* We denote *convolution* by $F \star G := m(F \boxtimes G)$. Further, we have

$$\begin{aligned} \Delta\colon D^b(\mathcal{M}_{\alpha+\beta}) &\to D^b(\mathcal{M}_\alpha \times \mathcal{M}_\beta), \\ F &\mapsto p_! q^*(F)[\dim p], \end{aligned}$$

called *restriction.* These are (co)associative in an appropriate sense.

We want to restrict to a smaller, semisimple, subcategory of $D^b(\mathcal{M}_\alpha)$ build up from simple perverse sheaves. For this, define

$$\mathbb{1}_\alpha := \underline{k}_{E_\alpha}[\dim E_\alpha] \in D^b(\mathcal{M}_\alpha),$$

the constant perverse sheaf (which is G_α-equivariant). Now define the Lusztig sheaves as

$$L_{\alpha_1,\ldots,\alpha_n} := \mathbb{1}_{\alpha_1} \star \ldots \star \mathbb{1}_\alpha \in D^b(\mathcal{M}_{\alpha_1+\ldots+\alpha_n}).$$

As a consequence of the associativity of the operation $\star$ (and pullbacks of constant sheaves are constant), we find another description of the Lusztig sheaves used for computations later:

Proposition 3.1. *Using the proper map $q\colon E_{\alpha_1,\ldots,\alpha_n} \to E_{\alpha_1+\ldots+\alpha_n}$, we have*

$$L_{\alpha_1,\ldots,\alpha_n} \cong q_!(\underline{k}_{E_{\alpha_1,\ldots,\alpha_n}}[\dim E_{\alpha_1,\ldots,\alpha_n} + \sum_i \dim E_{\alpha_i}]).$$

By the decomposition theorem, the Lusztig sheaves are semisimple. Denote

$$\mathcal{P}_\alpha := \{F \mid F \text{ simple perverse subsheaf of } L_{\alpha_1,\ldots,\alpha_n} \text{ for } \alpha = \alpha_1 + \ldots + \alpha_n,\ \alpha_i = \varepsilon_i \text{ (basis vector)}\}.$$

Define $\mathcal{H}_\alpha$ to be the full subcategory of $D^b(\mathcal{M}_\alpha)$ additively generated by shifts of elements of $\mathcal{P}_\alpha$, and set

$$\mathcal{P}_Q := \coprod_{\alpha \in \mathbb{N}^I} \mathcal{P}_\alpha, \qquad \mathcal{H}_Q := \coprod_{\alpha \in \mathbb{N}^I} \mathcal{H}_\alpha, \qquad \text{the } \textit{Hall category}.$$

We adapt the point of view that $\mathcal{P}_Q$ is a basis for $\mathcal{H}_Q$.

4. Some properties

Here, we include some results about the functors m and Δ which will be used to describe the algebraic structure of the K-group of $\mathcal{H}_Q$ later.

Proposition 4.1. *The category $\mathcal{H}_Q$ is preserved by the functors m and Δ. Moreover, we have for $\alpha = \beta + \gamma$, $\alpha = \sum_{i \in I} \alpha_i$*

$$\Delta(\overset{\in \mathcal{H}_\alpha}{L_{\alpha_1,\ldots,\alpha_n}}) = \bigoplus_{\substack{(\beta_1,\ldots,\beta_m),(\gamma_1,\ldots,\gamma_p) \\ \sum \beta_i = \beta, \sum \alpha_i = \alpha \\ \beta_i + \gamma_i = \alpha_i}} \overset{\in \mathcal{H}_\beta}{L_{\beta_1,\ldots,\beta_m}} \boxtimes \overset{\in \mathcal{H}_\gamma}{L_{\gamma_1,\ldots,\gamma_p}}[d_{\beta,\gamma}]$$

Proof. Closure under m can be checked on simple objects and thus, by closure under subobjects, on the Lusztig sheaves, where $L_{\alpha_1,\ldots,\alpha_n} \star L_{\beta_1,\ldots,\beta_m} = L_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m}$ by definition. Closure under Δ is harder and requires another description of $\mathcal{M}_\alpha$ as a quotient. The formula for Δ is derived from

$$\Delta(\mathbb{1}_\alpha) = \mathbb{1}_\beta \boxtimes \mathbb{1}_\gamma[-\langle \beta, \gamma \rangle].$$ □

Proposition 4.2. *$\mathcal{P}_\alpha$ is closed under Verdier duality D.*

Proof. This follows from $D(\mathbb{1}_\alpha) = \mathbb{1}_\alpha$, and that m commutes with D (up to shift). This follows from p being smooth, and hence $p^*D = Dp^! = Dp^*[2\dim p]$, and q proper giving $q_!D = Dq_* = Dq_!$. $\square$

5. The geometric pairing

For defining $U_\nu^{\mathbb{Z}}(\mathfrak{n}^+)$, the pairing is crucial giving self-duality, with Serre relations generating the radical. On the Hall category $\mathcal{H}_Q$, we have a pairing defined by G-equivariant cohomology:

$$\{F, G\} := \sum_j \dim H^j_{G_\alpha}(F \otimes G, E_\alpha)\nu^j \in \mathbb{N}((\nu)),$$

for objects $F, G \in \mathcal{H}_\alpha$. Some properties are straightforward:

$$\{F, G\} = \{G, F\}, \qquad \{F + F', G\} = \{F, G\} + \{F', G\},$$
$$\{F[n], G\} = \nu^n \{F, G\}.$$

Note that $[n]$ defines a $\mathbb{N}((\nu))$-action on $\mathcal{H}_Q$ and that $\mathcal{H}_Q$ is $\mathbb{N}^I = K_0(\mathbf{Rep}^{\mathrm{fd}}(Q))$-graded by definition.

Proposition 5.1. *For all $F, F', G \in \mathcal{H}_Q$ we have*

$$\{m(F \boxtimes F'), G\} = \{F \boxtimes F', \Delta(G)\}.$$

This resembles a "bialgebra" pairing in this categorified situation.

Proof. (heuristic)

$$\begin{aligned}
\{m(F \boxtimes F'), G\} &= \sum_j \dim H^j_{G_\alpha}(q_! p^*(F \boxtimes F')[\dim p] \otimes G, E_\alpha)\nu^j \\
&= \sum_j \dim H^j_{G_\alpha}(p^*(F \boxtimes F')[\dim p] \otimes q^*G, E_\alpha)\nu^j \\
&= \sum_j \dim H^j_{G_\alpha}(F \boxtimes F'[\dim p] \otimes p_! q^*G, E_\alpha)\nu^j \\
&= \{F \boxtimes F', \Delta(G)\},
\end{aligned}$$

where we use the projection formula. One however needs to draw large diagrams, replacing the spaces $E_{\alpha,\beta}$ by other versions before being able to actually use it. $\square$

References

[Lus10] G. Lusztig, *Introduction to quantum groups*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2010. Reprint of the 1994 edition. MR2759715 (2011j:17028)

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